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B.Sc part III

Topic :→

Continuity of Functions of Two-variables.

Definition :→ A Function $u = f(x,y)$ is said to be, Continuous at a given Point (a,b) of its domain of definition if

$$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$$

Thus $f(x,y)$ is Continuous at (a,b) if

$$|f(x,y) - f(a,b)| < \epsilon \text{ for all } (x,y) \text{ Satisfying}$$

$$|x-a| < \delta, \ |y-b| < \delta$$

i.e, $f(x,y)$ is continuous at (a,b) if the Simultaneous limit $\lim_{\substack{x\to a\\ y\to b}} f(x,y)$ Exists and is Equal to its Functional value $f(a,b)$ at (a,b)

Theorem :→ To prove that the Continuity of $f(x,y)$ at (a,b) implies the continuity of $f(x,y)$ as a function of x and as a function of y at a and b respectively but not conversely.

Proof :→ Since $f(x,y)$ is Continuous at (a,b)

therefore, $\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$

i.e Corresponding to an any Small +ve ϵ we can find a $\delta > 0$ such that

$$|f(x,y) - f(a,b)| < \epsilon/2 \text{ When } |x-a| < \delta \text{ and } |y-b| < \delta.$$

Now if (x_1, b), (x_2, b) be any two points such that the range of x is an interval $(a-\delta, a+\delta)$ lying within the domain of definition, then

$$|f(x_1,b) - f(x_2,b)| = |\{f(x_1,b) - f(a,b)\} + \{f(a,b) - f(x_2,b)\}|$$

$$\leq |f(x_1,b) - f(a,b)| + |f(x_2,b) - f(a,b)|$$

$$< \epsilon/2 + \epsilon/2 = \epsilon$$

i.e $|f(x,b)-f(a,b)|<\varepsilon$ where $|x-a|<\delta$

Similarly

$$|f(a,y_1)-f(a,y_2)|<\varepsilon$$

or $|f(a,y)-f(a,b)|<\varepsilon$ where $|y-b|<\delta$

i.e $f(a,y)$ is also a continuous function of one variable y for $y=b$

[the above condition is only necessary but not sufficient]

Examples:→

Let $f(x,y)=0$ for $x=0$ or for $y=0$

$=1$ for $x\neq 0,\ y\neq 0$

Soln:→ $f(x,0)=\left[f(x,y)\right]_{y=0}=0=f(0,0)$

$|f(x,0)-f(0,0)|=0<\varepsilon$

Which shows that $f(x,0)$ is a continuous function of one variable for $x=0$

Also $f(0,y)=\left[f(x,y)\right]_{x=0}=f(0,0)$

Therefore, $f(0,y)$ is also a continuous function of one variable for $y=0$

But $|f(x,y)-f(0,0)|=1$ (Not $<\varepsilon$)

Hence $f(x,y)$ is not continuous at $(0,0)$

This shows that although $f(x,0)$ is continuous function of one variable x at $x=0$ and $f(0,y)$ is also continuous function of one variable y at $y=0$ but $f(x,y)$ is not necessarily continuous at $(0,0)$

Ex 2.

Show that the function

$$f(x,y) = \frac{xy^3}{x^2+y^6}, \quad x \neq 0\ y \neq 0$$

and $f(0,0) = 0$

is not continuous at $(0,0)$,

but that the function is continuous in x alone and in y alone at the origin.

Sol^n. → First of all we shall show that the simultaneous limit of $f(x,y)$ at $(0,0)$ does not exist.

Let $(x,y) \longrightarrow (0,0)$ through any line $y = mx$

we have

$$\lim_{x\to 0} \frac{x \cdot m^3 x^3}{x^2 + x^6 m^6} = \lim_{x\to 0} \frac{m^3 x^4}{x^2 + x^6 m^6}$$

$$= \lim_{x\to 0} \frac{m^3 x^4}{x^2(1+x^4 m^6)}$$

$$= \lim_{x\to 0} \frac{m^3 x^2}{1+x^4 m^6} = \frac{0}{1} = 0$$

Now let $(x,y) \to (0,0)$ through the curve $x = y^3$.

then

$$\lim_{y\to 0} \frac{y^3 \cdot y^3}{y^6+y^6} = \lim_{y\to 0} \frac{y^6}{2y^6} = \frac{1}{2}$$

Since the limit obtained by two different routes are different therefore, the simultaneous limit does not exist. and hence the function is not continuous at $(0,0)$

However, the function is continuous in x alone and in y alone at the origin. for, by putting either variable zero and then letting the other variable $\to 0$, we get the limiting value zero, which is the value of $f(0,0)$

Ex:->

Show that the function

$$f(x,y) = \frac{xy}{\sqrt{x^2+y^2}}, \quad \begin{matrix} x \neq 0 \\ y \neq 0 \end{matrix} \quad \text{and } f(0,0) = 0$$

is Continuous at the origin $(0,0)$

We have $\lim\limits_{\substack{x\to 0 \\ y\to 0}} f(x,y) = \lim\limits_{\substack{x\to 0 \\ y\to 0}} \frac{xy}{\sqrt{x^2+y^2}}$

Let $\epsilon > 0$ be given. Put $x = r\cos\theta$, $y = r\sin\theta$.

Then $\left| \frac{xy}{\sqrt{x^2+y^2}} - 0 \right| = \left| \frac{r\cos\theta \cdot r\sin\theta}{\sqrt{r^2\cos^2\theta + r^2\sin^2\theta}} \right|$

$$= \left| \frac{r^2\cos\theta\sin\theta}{r} \right| = |r\cos\theta\sin\theta|$$

$$= r\left| \frac{\sin 2\theta}{2} \right| \quad \text{As } \sin\theta\cos\theta = \frac{\sin 2\theta}{2}$$

$$= \frac{r}{2}$$

$$= \frac{\sqrt{x^2+y^2}}{2} < \epsilon.$$

if $\frac{x^2}{4} < \frac{\epsilon^2}{2}$, i.e. $|x| < \sqrt{2}\epsilon$

$\frac{y^2}{4} < \frac{\epsilon^2}{2}$. $\quad |y| < \sqrt{2}\epsilon$.

Therefore, $\lim\limits_{(x,y)\to 0} \frac{xy}{\sqrt{x^2+y^2}} = 0 = f(0,0)$

Thus $f(x,y)$ is Continuous at $(0,0)$